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THE SECULAR ACCELERATION OF THE MOON.

BY G. W. HILL.

In the Philosophical Transactions for 1853, Prof. J. C. Adams, of Cambridge University, England, showed that the values of the secular acceleration of the mean motion of the moon, obtained by Plana and Damoiseau, were erroneous, for the simple reason that these authors had, inadvertently, made the solar eccentricity constant throughout a certain portion of the investigation. This statement of Prof. Adams gave rise to an animated and prolonged controversy, the history of which will, no doubt, always possess much interest.

I propose to obtain here the coefficient of the term in the moon's mean motion involving the square of the solar eccentricity, supposed variable, to quantities of the order of the square of the sun's disturbing force, when the lunar eccentricity and inclination of orbit are neglected. The method employed has no novelty, having been used before by Mr. Donkin. But, at the end of the investigation, I have found that it is possible to do without an explicit development of *R* in a periodic series, and thus the treatment is, to a considerable degree, abbreviated.

Let ζ denote the mean longitude of the moon as affected by this secular inequality, and n_0 the mean motion at a given epoch taken as the origin of time, we propose to prove that, in the equation

$$\frac{d\zeta}{dt} = n = n_0[1+H(e'^2_0 - e'^2)],$$

the true value of H is

$$\frac{3}{2}\left(\frac{n'}{n_0}\right)^2 - \frac{3771}{64}\left(\frac{n'}{n_0}\right)^4.$$

Employing the method of variation of the elements, we have, for determining the four elements n, ζ, e and ω of the lunar orbit, these equations

$$\frac{dn}{dt} = -\frac{3}{\mu a^2}\frac{dR}{d\zeta}, \qquad \frac{de}{dt} = -\frac{na}{\mu e}\frac{dR}{d\omega},$$

$$\frac{d\zeta}{dt} = n - 2\frac{na^2}{\mu}\frac{dR}{da} + \frac{1}{2}\frac{nae}{\mu}\frac{dR}{de}, \qquad \frac{d\omega}{dt} = \frac{na}{\mu e}\frac{dR}{de}.$$

In writing them, all terms, multiplied by higher powers of e than the first, have been neglected, as they are not needed in obtaining H to the degree of accuracy we propose. It may be noted that R is taken with such a sign that $\frac{dR}{dx}$ denotes the force tending to increase x.

Since we need not retain any terms multiplied by the ratio $\frac{a}{a'}$, the value of R is

$$R = \tfrac{1}{4}n'^2 r^2 \frac{a'^3}{r'^3}[1+3\cos 2(\lambda-\lambda')],$$

where λ and λ' are the true longitudes of the moon and sun. The constant part of R is evidently the same as that of $\frac{1}{4}n'^2 a^2 \frac{a'^3}{r'^3}$ when we reject e^2, that is, it is equal to $\frac{1}{4}n'^2 a^2(1+\frac{3}{2}e'^2)$.

Considering first those terms in R which are independent of e (we need those multiplied by e only when taking account of the effects produced by the variations δe and $\delta\omega$), we see that the only terms in R which produce terms in $\frac{dn}{dt}$, and, consequently, can give rise to terms independent of sines or cosines of arguments in $\frac{d\zeta}{dt}$, have arguments of the form $2\zeta+\psi$, where ψ denotes an angle depending on the sun's mean motion. Hence, denoting any one of these terms of R by $n'^2 a^2 A\cos(2\zeta+\psi)$, where A is independent of the lunar elements, but will generally contain e'^2, and regard being had to this term alone, the equations determining the elements become

$$\frac{dn}{dt} = 6n'^2 A \sin(2\zeta+\psi),$$

$$\frac{d\zeta}{dt} = n - 4\frac{n'^2}{n}A\cos(2\zeta+\psi),$$

where μ has been eliminated by using the equation $\mu = n^2 a^3$. Integrating these, and considering ψ as constant, since its variability affects only the terms in H multiplied by $\frac{n'^5}{n_0{}^5}$, but regard being had to the variability of A through e'^2, where we may consider $\frac{d.e'^2}{dt}$ as constant, we obtain

$$\delta n = -3\frac{n'^2}{n}A\cos(2\zeta+\psi)+\frac{3}{2}\frac{n'^2}{n^2}\frac{dA}{d.e'^2}\frac{d.e'^2}{dt}\sin(2\zeta+\psi),$$

$$\delta\zeta = -\frac{7}{2}\frac{n'^2}{n^2}A\sin(2\zeta+\psi)-\frac{5}{2}\frac{n'^2}{n^3}\frac{dA}{d.e'^2}\frac{d.e'^2}{dt}\cos(2\zeta+\psi).$$

This being only a first approximation in which we have had regard only to quantities of the order of n'^2, we proceed to a second approximation. And first, in the expression for $\frac{dn}{dt}$, we substitute for ζ, $\zeta+\delta\zeta$; and we find, for that part of the increment which is independent of the sines or cosines of arguments, the expression

$$\delta\cdot\frac{dn}{dt}=-15\frac{n'^4}{n^3}A\,\frac{dA}{d.e'^2}\,\frac{d.e'^2}{dt}.$$

Integrating this and putting $e'^2-e'^2_0=\delta.e'^2$,

$$\delta n\quad=-\frac{15}{2}\,\frac{n'^4}{n^3}\,\frac{d.A^2}{d.e'^2}\,\delta.e'^2.$$

Again, in the expression for $\frac{d\zeta}{dt}$, increasing n and ζ by their variations δn and $\delta\zeta$,

$$\delta.\frac{d\zeta}{dt}=-\frac{15}{2}\,\frac{n'^4}{n^3}\,\frac{d.A^2}{d.e'^2}\,\delta.e'^2-6\frac{n'^4}{n^3}A^2-14\frac{n'^4}{n^3}A^2.$$

Now the constant part of this value of $\delta.\frac{d\zeta}{dt}$ goes to form part of the constant n_0, hence, desiring to retain only the varying part, we may write $\frac{d.A^2}{d.e'^2}\,\delta.e'^2$ for A^2, and thus obtain

$$\delta.\frac{d\zeta}{dt}=-\frac{55}{2}\,\frac{n'^4}{n^3}\,\frac{d.A^2}{d.e'^2}\,\delta.e'^2. \tag{1}$$

In the next place let us consider the terms in R multiplied by e; they are all of the form

$$n'^2a^2eA\cos(\omega+k\zeta+\psi),$$

where A and ψ possess the same quality as before, and k may be -3, -1 or 1. Representing $\omega+k\zeta+\psi$ by θ, the equations determining the elements are, regard being had to this term alone,

$$\frac{dn}{dt}=3kn'^2Ae\sin\theta,\qquad\qquad \frac{de}{dt}=\frac{n'^2}{n}A\sin\theta,$$

$$\frac{d\zeta}{dt}=n-\frac{7}{2}\frac{n'^2}{n}Ae\cos\theta,\qquad\qquad \frac{d\theta}{dt}=\frac{n'^2}{n}A\,\frac{1}{e}\cos\theta.$$

In the last equation we have written only the term divided by e, since this alone can produce terms in $\delta.\frac{d\zeta}{dt}$ of the kind we seek. Integrating the last two as we integrated in the former case, we obtain

$$\delta e=-\frac{1}{k}\,\frac{n'^2}{n^2}A\cos\theta+\frac{1}{k^2}\,\frac{n'^2}{n^3}\,\frac{d.A^2}{d.e'^2}\,\frac{d.e'^2}{dt}\sin\theta,$$

$$\delta\theta=\frac{1}{ke}\,\frac{n'^2}{n^2}A\sin\theta+\frac{1}{k^2e}\,\frac{n'^2}{n^3}\,\frac{d.A^2}{d.e'^2}\,\frac{d.e'^2}{dt}\cos\theta.$$

Augmenting, in the expression for $\frac{dn}{dt}$, e and θ by these quantities, we obtain, regard being had only to the terms which are independent of sines or cosines of angles,

$$\delta.\frac{dn}{dt}=\frac{3}{2k}\frac{n'^4}{n^3}\frac{d.A^2}{d.e'^2}\frac{d.e'^2}{dt}.$$

Integrating this we have

$$\delta n=\frac{3}{2k}\frac{n'^4}{n^3}\frac{d.A^2}{d.e'^2}\delta.e'^2.$$

Increasing in the expression for $\frac{d\zeta}{dt}$, the elements n, e and θ by their variations δn, δe and $\delta\theta$, and preserving only the terms independent of the sines or cosines of angles, we get

$$\delta.\frac{d\zeta}{dt}=\frac{3}{2k}\frac{n'^4}{n^3}\frac{d.A^2}{d.e'^2}\delta.e'^2+\frac{7}{2k}\frac{n'^4}{n^3}A^2.$$

In like manner as before, rejecting the constant part of this which coalesces with n_0, we obtain

$$\delta.\frac{d\zeta}{dt}=\frac{5}{k}\frac{n'^4}{n^3}\frac{d.A^2}{d.e'^2}\delta.e'^2. \qquad (2)$$

When formulæ (1) and (2) are applied to all the terms of R to which each is applicable and the results added we shall have the complete value of $\delta.\frac{d\zeta}{dt}$, since it is plain that the combination of two different terms in R will always produce terms in $\delta.\frac{d\zeta}{dt}$ involving the sines or cosines of angles.

Denoting the mean anomalies of the moon and sun by ξ and ξ', and the mean angular distance of these bodies by τ, the part of aR which is independent of e may be written

$$aR=A_0+A_1\cos 2\tau+A_2\cos\xi'+A_3\cos(2\tau-\xi')+A_4\cos(2\tau+\xi').$$

Formula (1), applied to this series, gives

$$\delta.\frac{d\zeta}{dt}=-\frac{55}{2}n\frac{d.(A_1^2+A_3^2+A_4^2)}{d.e'^2}\delta.e'^2.$$

We can obtain the terms in R multiplied by e from the series just given by using the equation

$$\frac{dR}{de}=r\frac{dR}{dr}\frac{d.\log r}{de}+\frac{dR}{d\lambda}\frac{d\lambda}{de}$$

$$=-2R\cos\xi+2\frac{dR}{d\tau}\sin\xi.$$

Whence

$$\begin{aligned}\frac{d.aR}{de}=&-2A_0\cos\xi-3A_1\cos(2\tau-\xi)+A_1\cos(2\tau+\xi)\\&-A_2\cos(\xi-\xi')-A_2\cos(\xi+\xi')\\&-3A_3\cos(2\tau-\xi'-\xi)+A_3\cos(2\tau-\xi'+\xi)\\&-3A_4\cos(2\tau+\xi'-\xi)+A_4\cos(2\tau+\xi'+\xi).\end{aligned}$$

Applying formula (2) to this series gives

$$\delta.\frac{d\zeta}{dt}=n\,\frac{d}{d.e'^2}[-5(4A_0^2+A_2^2+A_2^2)-\tfrac{5}{3}(A_1^2+A_3^2+A_4^2)$$
$$+5(9A_1^2+9A_3^2+9A_4^2)]\,\delta.e'^2$$

$$=n\,\frac{d}{d.e'^2}\left[\frac{130}{3}(A_1^2+A_3^2+A_4^2)-10(2A_0^2+A_2^2)\right]\delta.e'^2.$$

Adding this to the expression given by formula (1),

$$\delta.\frac{d\zeta}{dt}=n\,\frac{d}{d.e'^2}\left[\frac{95}{6}(A_1^2+A_3^2+A_4^2)-10(2A_0^2+A_2^2)\right]\delta.e'^2.$$

But, denoting the constant term of a^2R^2 by K, we have

$$K=A_0^2+\tfrac{1}{2}(A_1^2+A_2^2+A_3^2+A_4^2).$$

Or

$$A_1^2+A_3^2+A_4^2=2K-(2A_0^2+A_2^2),$$

and

$$\delta.\frac{d\zeta}{dt}=n\,\frac{d}{d.e'^2}\left[\tfrac{95}{3}K-\tfrac{155}{6}(2A_0^2+A_2^2)\right]\delta.e'^2.$$

But we have

$$a^2R^2=\frac{1}{16}\,\frac{n'^4}{n^4}\,\frac{a'^6}{r^6}\left[\tfrac{11}{2}+6\cos 2(\lambda-\lambda')+\tfrac{9}{2}\cos 4(\lambda-\lambda')\right],$$

and hence K is equal to the constant term of $\frac{11}{32}\,\frac{n'^4}{n^4}\,\frac{a'^6}{r'^6}$. In consequence, denoting the constant term of $\frac{a'^6}{r'^6}$ by L, we shall have

$$\delta.\frac{d\zeta}{dt}=n\,\frac{d}{d.e'^2}\left[\frac{1045}{96}\,\frac{n'^4}{n^4}L-\frac{155}{6}\left(2A_0^2+A_2^2\right)\right]\delta.e'^2.$$

Also we evidently have

$$\frac{1}{4}\,\frac{n'^2}{n^2}\,\frac{a'^3}{r'^3}=A_0+A_2\cos\xi',$$

and thus

$$\frac{1}{8}\,\frac{n'^4}{n^4}\,L=2A_0^2+A_2^2.$$

Substituting this value

$$\delta.\frac{d\zeta}{dt}=n\,\frac{d}{d.e'^2}\left[\frac{245}{32}\,\frac{n'^4}{n^4}\,L\right]\delta.e'^2.$$

But the constant term of $\frac{a'^6}{r'^6}$ is known to be $1+\frac{15}{2}e'^2$; hence, in fine,

$$\delta.\frac{d\zeta}{dt}=\frac{3675}{64}\,\frac{n'^4}{n^3}\,\delta.e'^2.$$

To obtain $\frac{d\zeta}{dt}$ we must add to n both this term and that which arises, in the first aporoximation, from the term $-\frac{2na^2}{\mu}\frac{dR}{da}$ in the differential equation for $\frac{d\zeta}{dt}$, which is therefore equal to the constant term of $-\frac{n'^2}{n}\frac{a'^3}{r'^3}$, that is, to $-\frac{n'^2}{n}\left(1+\frac{3}{2}e'^2\right)$. Thus

$$\frac{d\zeta}{dt}=n-\frac{n'^2}{n}\left(1+\frac{3}{2}e'^2_0\right)-\left(\frac{3}{2}\frac{n'^2}{n}-\frac{3675}{64}\frac{n'^4}{n^3}\right)\delta.e'^2.$$

We could have added to the first two terms of this equation a term $B\frac{n'^4}{n^3}e'^2_0$, where B is a numerical coefficient, equal to the aggregate of the constants we have virtually neglected whenever we wrote $\delta\,.\,e'^2$ for e'^2, but it will be easily seen that this would not change the final result. We evidently have

$$n_0=n-\frac{n'^2}{n}\left(1+\frac{3}{2}e'^2_0\right).$$

From which, to a sufficient degree of approximation,

$$n=n_0+\frac{n'^2}{n_0}.$$

Substituting this value of n, we get

$$\frac{d\zeta}{dt}=n_0\left[1+\left(\frac{3}{2}\frac{n'^2}{n_0^2}-\frac{3771}{64}\frac{n'^4}{n_0^4}\right)\left(e'^2_0-e'^2\right)\right].$$

DISCUSSION OF THE GENERAL EQUATION OF THE THIRD DEGREE.

BY JOHN BORDEN, CHICAGO, ILL.

$$x^3+Ax^2+Bx+C=y, \tag{1}$$

$$3x^2+2Ax+B=\frac{dy}{dx}, \tag{2}$$

$$6x+2A=\frac{d^2y}{dx^2}. \tag{3}$$

If $\frac{dy}{dx}=0$, then, in (2), $$x=-\frac{A}{3}\pm\sqrt{\left(-\frac{B}{3}+\frac{A^2}{9}\right)}; \tag{4}$$

and if $\frac{d^2y}{dx^2}=0$, then, in (3), $$x=-\frac{A}{3}. \tag{5}$$

First Case; A and B positive and $B<\frac{1}{3}A^2$.—The form of the curve is as in the Fig.; its locus being taken for $C=0$.